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On the pseudomomentum and generalized Stokes drift in a spectrum of rotational waves

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Two Lagrangian-mean measures crucial to the accurate estimation of mean particle velocities in wavy or turbulent shear flows are considered. The measures are the pseudomomentum and generalized Stokes drift and of particular interest is their expression in terms of quantities directly measurable by fixed instruments. To proceed, the measures are first calculated for broad spectra of progressive symmetric rotational wave pairs of small amplitude. Both discrete and continuous spectra are considered and the waves may grow or decay. The expressions are then cast into a form composed of quantities that are measurable in a fixed reference frame, such as the surface slope spectrum of surface gravity waves or space–time velocity correlations in the interior of wavy shear flows. Finally, an example is given in which the measures are calculated for a plane channel flow subject to a broad spectrum of discrete progressive waves, specifically a numerical simulation of turbulent channel flow. It is seen that while the streamwise component of pseudomomentum is everywhere negative in the flow, the generalized Stokes drift changes sign, giving rise to an enhanced mass transport close to the boundary and a reduction in transport some distance from it. The sign change occurs 12.5 viscous units from the wall, near the centre of a 15 viscous units thick highly sheared layer of Stokes drift.

1. Introduction

Mean particle velocities in wavy and turbulent shear flows differ from the mean Eulerian flow, yet while the latter is directly measurable by fixed instruments, the former Lagrangian mean is not. Nevertheless, it is important to be able to make credible estimates of mean particle transport, be it in the shear flow beneath surface waves or in the turbulent flow near a rigid boundary. This paper is concerned with the difficult question of how to express specific Lagrangian mean quantities, namely the pseudomomentum and generalized Stokes drift, in terms of quantities directly measurable by fixed instruments. Only by achieving this can credible estimates then be made of the mean particle velocity in wavy or turbulent flow and thus of how it differs from the mean Eulerian velocity.

Formally, the pseudomomentum and generalized Stokes drift are measures of the nonlinear interaction of the fluctuations with themselves and supporting shear flow. We calculate them for broad spectra of small-amplitude progressive rotational $O(\epsilon)$ waves. The measures arise in Andrews & McIntyre's (1978, hereinafter referred to as AM) generalized Lagrangian mean (GLM) formulation, which is an exact theory of nonlinear waves on a Lagrangian mean flow. GLM is compelling because it describes Lagrangian aspects of fluid motion from an Eulerian framework; a feature that has led to its use in studies ranging from the interaction of internal and inertial waves (Broutman & Grimshaw 1988) to transport processes in oceanic (Gent *et al.* 1995) and atmospheric chemistry (Mahlman 1997), to the dynamics of barotropic storm tracks (Swanson, Kushner & Held 1997).

Also compelling is that GLM-theory describes mean vorticity kinematics in the same way as instantaneous vorticity kinematics are described; this enables it to capture structural details, in contrast to Reynolds or Reynolds–Hussain (Reynolds & Hussain 1970) averaging which masks vorticity kinematics. As an avenue to elucidate structures which arise in wavy shear flows, therefore, the GLM formulation is canonical; indeed, not only does it recover the Craik–Leibovich (Craik & Leibovich 1976) equations when the wave field is irrotational and the shear is weak (Leibovich 1980), but it makes possible the extension of Craik–Leibovich instability theory when the wave field is rotational and the shear strong (Craik 1982*c*). Instability of the wave-mean interaction is determined, in this instance, by the Craik–Phillips–Shen criterion (Craik 1982*c*; Phillips & Shen 1996) and the structures that arise are longitudinal vortices or eddies (see Craik 1982*c*; Phillips & Wu 1994; Phillips, Wu & Lumley 1996).

However, like the Reynolds-averaged equations, the GLM equations are not closed; closure requires the aforementioned measures. These, of course, are readily calculable if the wave field is monotonic or irrotational, but that is not always the case and to make further progress with instability studies of the type pioneered by Craik, or, say, in estimating transport characteristics of pollutants in solution or suspension (Bratseth 1998), we require expressions for the aforementioned nonlinear measures in a form applicable to any spectrum of waves in a shear flow. Of course, the wavefield and shear flow cannot be specified arbitrarily, they must together be a solution to the instantaneous equations of motion, which necessitates a direct simulation prior to calculating the nonlinear measures.

Alternatively, the measures could be acquired in an experiment. Unfortunately, direct measurements are unlikely because mean particle velocities or other Lagrangian averages are not recordable by fixed instruments such as hot-wire or laser-Doppler anemometers (although particle image velocimetry may be an exception). Nevertheless such instruments can record Reynolds stresses directly and these are analogous to the Lagrangian averages or measures we seek.

The object of this paper, therefore, is twofold: first, to obtain expressions for the measures, specifically the generalized Stokes drift and pseudomomentum, that apply to any spectrum of rotational waves in a shear flow; and secondly, to cast those expressions in terms of quantities that are 'measurable' in an Eulerian frame, where 'measurable' means recordable in an experiment by fixed instruments or calculable numerically in an Eulerian-field direct simulation.

The importance of the first measure was identified by Rayleigh (1896) as the cause for acoustic streaming, but it is equally widely known in the context of propagating surface gravity (or capillary gravity) waves, where it is manifest as a mean drift velocity in the direction of wave propagation. In this instance it is denoted the Stokes drift, after Stokes (1847) who predicted it in the absence of mean shear assuming inviscid theory and irrotational waves. When a sheared mean flow is present, however, and the waves are rotational, the term 'generalized Stokes drift' $\boldsymbol{d}$ is used.

Interestingly, the fluctuating particle motions arising from this $O(\epsilon^2)$ nonlinearity induce no mean Eulerian flow in ideal fluids, but do induce such flow in real fluids. Indeed, Longuet-Higgins (1953) established that a second-order mean vorticity is generated in the viscous boundary layer at a free surface (and channel bottom) and that as its vorticity is diffused, it induces in the interior a non-zero mean Eulerian current $\bar{\boldsymbol{u}}$, whose magnitude is greatly affected by surface contamination (Craik 1982*a*). Physically, the ensuing mass transport velocity, or Lagrangian mean velocity $\bar{\boldsymbol{u}}^L$, is the sum of $\bar{\boldsymbol{u}}$ and $\boldsymbol{d}$.

Of course this Eulerian current can be further enhanced by external means, such as wind shear or pressure gradients, but whatever details determine $\bar{\boldsymbol{u}}$, it is desirable to describe any ensuing wave–mean flow interactions by a set of equations that depict $\bar{\boldsymbol{u}}^L$ as the dependent variable. Such equations should also possess conservative properties analogous to those of the instantaneous Navier–Stokes equations. Following much effort, this aim was realized in the GLM equations of Andrews & McIntyre; and inherent therein is the second measure of nonlinear interaction, the pseudomomentum $\boldsymbol{p}$. Interestingly, although $\boldsymbol{p}$ and $\boldsymbol{d}$ are not in general equal, their irrotational components concur to $O(\epsilon^4)$ (see AM § 6; Craik 1982*c*), so that only one measure plays a role when the wave field is irrotational.

Stokes' expression for a single wavetrain in deep water (see also O.M. Phillips 1966), was extended to a discrete symmetric spectrum of irrotational water waves of equal amplitude by Craik & Leibovich (1976), and to a random field of irrotational surface gravity-waves by Kenyon (1969) and Huang (1971). Craik (1982*b*, 1985) was the first to allow for rotational waves and gives general expressions for the pseudomomentum and generalized Stokes drift in a single train of two-dimensional linear $O(\epsilon)$ waves in the presence of shear. Attempts to do likewise for statistically stationary fluctuating fields that exhibit continuous spectra were made by Lumley (1986), Phillips (1988) and Leibovich (1992), but the ensuing expressions depict behaviour which is divergent and potentially oscillatory in time. That such features can occur had earlier been foreseen by Craik (1982*b*), who notes that they result from an averaged ensemble of particles initially located on different streamlines or when particles on the same streamline are unevenly distributed between peaks and troughs. The best way to circumvent such features, which play no role in the interpretation of the Lagrangian mean velocity, is to begin the average when the wave amplitude is effectively zero and all particles are evenly spaced along a streamline of the flow.

Our intent here is to derive general expressions for the pseudomomentum and generalized Stokes drift which contain only those elements which contribute directly to the Lagrangian mean velocity; and, in particular, expressions that are valid for two or three dimensional discrete or continuous spectra of $O(\epsilon)$ rotational waves in the presence of shear. With such information at hand, we then cast the expressions in terms of measurable quantities, obtainable either from direct numerical simulation or experimentally. We begin in § 2 with an outline of GLM and in § 3 consider a discrete spectrum of symmetric $O(\epsilon)$ rotational wave pairs in strong shear. In § 4 we discuss measurable quantities and in § 5 generalize our results to a continuous spectrum of waves. An example is given in § 6 and the work is discussed in § 7.

2. The generalized Lagrangian-mean formulation

2.1. *Background*

The generalized Lagrangian mean equations in AM are a mapping of Navier–Stokes into a material frame in which the analogy of mean vorticity is conserved. In consequence, the equations provide a very general Lagrangian-mean description of the back effect of oscillatory disturbances upon the mean state, and depict a Lagrangian-mean velocity field that describes trajectories about which fluctuating particle motions have zero mean, when any averaging process, be it temporal, spatial, ensemble or other, is applied. Moreover, provided the mapping is invertible, the equations are exact and thus valid for waves of all amplitudes, although for practical purposes they have so far been restricted to waves of small amplitude, measured by a dimensionless parameter ϵ, so that any displacement $\boldsymbol{\xi}$ from the mean trajectory is $O(\epsilon)$ compared to the wavelength of the wave field.

In order to define an exact Lagrangian-mean operator $(\bar{\ })^L$, corresponding to any given Eulerian-mean operator $(\bar{\ })$, it is necessary to define with equal generality an exact, disturbance-associated particle displacement field $\boldsymbol{\xi}(\boldsymbol{x},t)$. Then, for any scalar or tensor field, φ say, of any rank, it is possible to introduce the mapping $\boldsymbol{x} \mapsto \boldsymbol{x}+\boldsymbol{\xi}$ and write

$$\overline{\varphi(\boldsymbol{x},t)}^L = \overline{\varphi^{\xi}(\boldsymbol{x},t)} \quad \text{where} \quad \varphi^{\xi}(\boldsymbol{x},t) = \varphi(\boldsymbol{x}+\boldsymbol{\xi},t).$$

Then, on choosing a GLM such that $\overline{\boldsymbol{\xi}(\boldsymbol{x},t)} = 0$, there is, for any given Eulerian velocity $\boldsymbol{u}(\boldsymbol{x},t)$, a unique Lagrangian-mean velocity, $\bar{\boldsymbol{u}}^L$, which is related to the Eulerian-mean velocity by the generalized Stokes drift $\boldsymbol{d}$, as $\bar{\boldsymbol{u}}^L = \bar{\boldsymbol{u}} + \boldsymbol{d}$. Furthermore, in terms of the Lagrangian-mean material derivative, $\bar{D}^L = \partial/\partial t + \bar{\boldsymbol{u}}^L \cdot \nabla$, it then follows that

$$\bar{D}^L \boldsymbol{\xi} = \boldsymbol{u}^{\ell}, \tag{2.1}$$

where the Lagrangian disturbance velocity $\boldsymbol{u}^{\ell}$ is given by $\boldsymbol{u}^{\ell}(\boldsymbol{x},t) = \boldsymbol{u}^{\xi} - \bar{\boldsymbol{u}}^L$, such that $\bar{\boldsymbol{u}}^{\ell} = 0$.

Cogent, but somewhat different, outlines of the derivation of the GLM equations are given by Craik (1985) and Leibovich (1992), with complete details in AM. Specifically, for homentropic flows of constant density ρ in a non-rotating reference frame, the GLM momentum equation is:

$$\bar{D}^L(\bar{u}_l^L - p_l) + \bar{u}_{k,l}^L(\bar{u}_k^L - p_k) + \Pi_{,l} = \bar{\mathscr{X}}_l,$$

$$\Pi = \frac{\bar{\wp}}{\rho} + \bar{\Phi}_l^L - \tfrac{1}{2}\overline{u_j^{\xi} u_j^{\xi}}.$$

Here, repeated indices imply summation and commas denote partial differentiation; furthermore, Φ is the force potential per unit mass, $\mathscr{X}$ is a function which allows for dissipative forces and $\wp$ is pressure.

Of interest in the present work is the term responsible for nonlinear forcing of the mean flow, the vector wave property $\boldsymbol{p}$, whose lth component is

$$p_l = -\overline{\xi_{j,l} u_j^{\ell}}. \tag{2.2}$$

Physically, the vector $\boldsymbol{p} = p_l(\boldsymbol{x},t)$ is a measure of the nonlinear interaction of the waves both with themselves and the mean flow; it is denoted the pseudomomentum or quasi-momentum per unit mass (McIntyre 1988).

2.2. *Small-amplitude waves*

Our intent is to express p_l in terms of Eulerian correlations (velocity or other) that are measurable experimentally in either discrete or continuous spectra of small-amplitude rotational waves. For generality, we assume the waves occur in a shear flow, with which they interact. However, in following such interactions with GLM, we must be cautious that the mapping from the true Lagrangian to the reference generalized Lagrangian mean remains invertible. Since this condition is reflected by the Jacobian J and fails when $J = 0$ it is prudent to monitor the temporal behaviour of J. We thus begin by calculating the Jacobian which, for incompressible, Boussinesq flows in which ϵ is characteristic of the initial disturbance, takes the form (AM)

$$J = 1 - \tfrac{1}{2}\overline{(\xi_j \xi_k)}_{,jk} + O(\epsilon^3). \tag{2.3}$$

Also of interest is the generalized Stokes drift

$$d_l = \overline{\xi_j \breve{u}_{l,j}} + \tfrac{1}{2}\overline{\xi_j \xi_k}\bar{u}_{l,jk} + O(\epsilon^3), \tag{2.4}$$

which, because the Eulerian fluctuating velocity is $\breve{\boldsymbol{u}} = \boldsymbol{u}(\boldsymbol{x}, t) - \bar{\boldsymbol{u}}(\boldsymbol{x}, t)$, leads to an expression for the small-amplitude Lagrangian velocity perturbation as

$$u_j^\ell = \breve{u}_j + \xi_k \bar{u}_{j,k} + O(\epsilon^2), \tag{2.5}$$

thereby permitting the pseudomomentum (2.2) to be written as

$$-p_l = \overline{\xi_{j,l}\breve{u}_j} + \overline{\xi_{j,l}\xi_k}\bar{u}_{j,k} + O(\epsilon^3). \tag{2.6}$$

Of course, to evaluate (2.4) and (2.6) we require the displacement field, given the wave field. To proceed, we note that $\bar{D}^L \xi_j = \mathrm{d}\xi_j/\mathrm{d}t$ and employ (2.1) and (2.5); then $\xi_j(\boldsymbol{x}, t)$ is given by integration of

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = \breve{u}_j + \xi_k \bar{u}_{j,k} \tag{2.7}$$

along mean trajectories

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \bar{\boldsymbol{u}}^L(\boldsymbol{x}, t).$$

Unfortunately, evaluating (2.7) is not in general straightforward. However, provided that $\boldsymbol{\xi}$ is small compared with the radius of curvature of $\bar{\boldsymbol{u}}$, then $\bar{\boldsymbol{u}}$ may be treated as constant for the purposes of integration; and that is the case here, where we envisage the flow to be predominantly in the x-direction and to be a function of z. Then, subject to postulate *viii* of AM's GLM formulation that ξ_j vanish at $\boldsymbol{x} = \boldsymbol{x}_0$, $t = t_0$, we find

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = \breve{u}_j + \xi_3 \bar{u}_{1,3}\delta_{j1}$$

on, say, $\boldsymbol{x} = \boldsymbol{x}_0 + \bar{\boldsymbol{u}}t + O(\epsilon^2)$, so that

$$\xi_j(\boldsymbol{x}, t) = \int_{t_0}^{t} \breve{u}_j[\boldsymbol{x}(\zeta), \zeta]\mathrm{d}\zeta + \delta_{j1}\bar{u}_{1,3}\int_{t_0}^{t} \xi_3[\boldsymbol{x}(\zeta), \zeta]\mathrm{d}\zeta. \tag{2.8}$$

3. A discrete spectrum of waves

Consider a small but finite-amplitude three-dimensional disturbance defined by a discrete spectrum of wavenumbers in a parallel shear flow $\bar{\boldsymbol{u}} = U(z)\boldsymbol{i}$ of constant density. Then,

$$\boldsymbol{u}(x, y, z, t) = [U(z) + \breve{u}_1, \breve{u}_2, \breve{u}_3], \tag{3.1}$$

where $\breve{\boldsymbol{u}}(x, y, z, t)$ is a disturbance that satisfies the continuity equation and is a solution to the Navier–Stokes equation subject to (3.1) and relevant boundary conditions; for example, plane rigid boundary conditions at $z = 0, H$ say, or a rigid boundary at $z = -H$ with a free surface or fluid–fluid interface at $z = 0$. Of interest are the pseudomomentum and generalized Stokes drift when $\nabla \times \breve{\boldsymbol{u}} \neq 0$; the case $\nabla \times \breve{\boldsymbol{u}} = 0$ was considered by Craik & Leibovich (1976).

We confine attention to flows periodic in x and note that solutions for $\breve{\boldsymbol{u}}$ strictly periodic in x at some instant t remain so for all time. We also assume H is finite (we shall allow for $H \to \infty$ in § 5). Then, the temporal eigenvalue spectrum of the linear operator acting on $\breve{\boldsymbol{u}}$ indicates that for each Fourier component $\exp(\mathrm{i}k\alpha x)\hat{\boldsymbol{u}}_{kl}(z,t)[\cos l\beta y, \sin l\beta y, \cos l\beta y]$ $(k, l = 0, \pm1, \pm2, \ldots)$ of $\breve{\boldsymbol{u}}$ at fixed Reynolds number R, there exists a complete set of discrete eigenfunctions $\phi^n_{kl}(z)$ and $\mho^n_{kl}(z)$ with eigenvalues ω^n_{kl} for $n = 1, 2, \ldots, \infty$ (Lin 1961). In particular, for symmetric wave pairs

$$\hat{\boldsymbol{u}}_{kl}(z,t) = \exp(-\mathrm{i}\omega^n_{kl}t)\left[\frac{k^2\alpha^2}{\gamma^2}\phi^{n\,\prime}_{kl} - \frac{l^2\beta^2}{\gamma^2}\mho^n_{kl}, \frac{\mathrm{i}k\alpha l\beta}{\gamma^2}(\phi^{n\,\prime}_{kl} + \mho^n_{kl}), -\mathrm{i}k\alpha\phi^n_{kl}\right]$$
$$(k, l = 0, \pm1, \pm2, \ldots) \quad (n = 1, 2, \ldots, \infty), \tag{3.2}$$

where α and β are fixed wavenumbers in the streamwise and spanwise directions and prime denotes $\mathrm{d}/\mathrm{d}z$.

Here, $\phi^n_{kl}(z)$ is the nth Orr–Sommerfeld eigenfunction for wavenumber $\gamma = (k^2\alpha^2 + l^2\beta^2)^{1/2}$ which satisfies

$$[-\mathrm{i}\gamma U\Delta + \mathrm{i}\gamma U'' + R^{-1}\Delta\Delta]\phi = -\mathrm{i}\omega\Delta\phi, \tag{3.3a}$$

subject to appropriate boundary conditions, while its counterpart $\mho^n_{kl}(z)$ satisfies (as a forced response) the vertical vorticity equation

$$[-\mathrm{i}\gamma U + R^{-1}\Delta]\mho - \mathrm{i}U'\phi = -\mathrm{i}\omega\mho \tag{3.3b}$$

in which $\Delta \equiv \mathrm{d}^2/\mathrm{d}z^2 - \gamma^2$ (see also Craik 1970; Gustavsson & Hultgren 1980; Butler & Farrell 1992).

In consequence, disturbances resulting from finite-amplitude waves may be formally expanded as $(j = 1, 2, 3)$

$$\breve{u}_j(x,y,z,t) = \epsilon\mathrm{Re}\left\{\exp(\mathrm{i}\varpi^n_{kl}(t))E^n_{jkl}(z,t)\cos[l\beta(y+y_l) - \delta_{j2}\tfrac{1}{2}\pi]\right\}$$
$$(k, l = 0, \pm1, \pm2, \ldots) \quad (n = 1, 2, \ldots, \infty) \tag{3.4}$$

where repeated indices imply summation, $\varpi^n_{kl}(t) = k\alpha x - {\omega^0}^n_{kl}t$ and y_l is a spanwise offset that varies randomly with the counter l, with (from (3.2)),

$$E^n_{jkl} = A^n_{kl}(t)\left[\frac{k^2\alpha^2}{\gamma^2}\phi^{n\,\prime}_{kl}(z) - \frac{l^2\beta^2}{\gamma^2}\mho^n_{kl}(z), \frac{\mathrm{i}k\alpha l\beta}{\gamma^2}\left(\phi^{n\,\prime}_{kl}(z) + \mho^n_{kl}(z)\right), -\mathrm{i}k\alpha\phi^n_{kl}(z)\right].$$

Here, the complex amplitude (at $k\alpha$, $l\beta$ and n) is $\epsilon A^n_{kl}(t)$, where ϵ is characteristic of the wave slope of the complete disturbance, and the eigenfunctions $\phi^n_{kl}(z)$ and $\mho^n_{kl}(z)$ are unity-normalized. Given boundary conditions and details of the problem to hand, e.g. water depth, fluid properties, stratification, etc., a specific equation can be derived to define the complex amplitude (see Craik 1985 § 18). However, to ensure real physical disturbances, it is necessary always that $A^n_{-k(-l)} = {A^*}^n_{kl}$, where $*$ indicates complex conjugate. Such analyses also relate temporal modulations of wave amplitude to the wave period, typically with the scaling $\kappa = \epsilon^\lambda t$ with $\lambda \geqslant 0$. Of course, the actual value of λ is dependent upon boundary conditions and features pertinent to the problem; for example, λ is typically 2 for amplitude modulations of weakly nonlinear surface waves in inviscid fluid.

Finally, $\omega^{0^n}_{kl} = \mathrm{Re}\{\omega^n_{kl}\}$ is the real part of the nth root of the linear dispersion equation. To reduce clutter, however, we shall drop the n and write ω^0_{kl}, noting that although we retain all n for generality, it is reasonable, for most purposes, to restrict attention to the least damped $(n = 0)$ modes. Indeed, if the wavepacket (3.3b) is weakly nonlinear and centred on a single wavenumber and frequency, then ω^0_{kl} will reduce to a particular real root (of the linear dispersion equation) that is characteristic of the wavenumber and the shear flow under consideration.

Turning now to (2.8), and writing $Y = y + y_l$, we see that the displacements take the form

$$\xi_j(x, y, z, t) = \epsilon \mathrm{Re}\left\{\exp(\mathrm{i}\varpi^n_{kl}(t))G^n_{jkl}(z, t)\cos(l\beta Y - \delta_{j2}\tfrac{1}{2}\pi)\right\}, \tag{3.5}$$

where

$$G^n_{1kl} = I^n_{kl}(z, t)\left(\frac{k^2\alpha^2}{\gamma^2}\phi^{n\,\prime}_{kl}(z) - \frac{l^2\beta^2}{\gamma^2}\mho^n_{kl}(z)\right) - \mathrm{i}k\alpha U'(z)K^n_{kl}(z, t)\phi^n_{kl}(z),$$

$$G^n_{2kl} = \frac{\mathrm{i}k\alpha l\beta}{\gamma^2}I^n_{kl}(z, t)\left(\phi^{n\,\prime}_{kl}(z) + \mho^n_{kl}(z)\right), \quad G^n_{3kl} = -\mathrm{i}k\alpha I^n_{kl}(z, t)\phi^n_{kl}(z),$$

with the Lagrangian integrals

$$I^n_{kl} = \exp(-\mathrm{i}\varpi^n_{kl}(t))\int_{t_0}^{t} A^n_{kl}(\zeta)\exp(\mathrm{i}\varpi^n_{kl}(\zeta))\mathrm{d}\zeta \tag{3.6}$$

and

$$K^n_{kl} = \exp(-\mathrm{i}\varpi^n_{kl}(t))\int_{t_0}^{t}\int_{t_0}^{p} A^n_{kl}(\zeta)\exp(\mathrm{i}\varpi^n_{kl}(\zeta))\mathrm{d}\zeta\,\mathrm{d}p. \tag{3.7}$$

3.1. *Averaging*

In view of its importance *vis à vis* the invertibility of the mapping to GLM, we look first at the Jacobian (2.3), whose portion to be averaged has the form $\xi_j[\boldsymbol{x}(t), t]\xi_l[\boldsymbol{x}(t), t]$. Observe that it is evaluated at one instant in time t and thus at one point along the mean trajectory $x(t)$, as are all averages in the GLM formulation. For the sake of generality, however, we shall assume that the components of all such correlations are separated in time, say at t and s and thus evaluated at points $x(t)$ and $x(s)$ along the mean trajectory. The reason for doing so will not be apparent until later in the analysis (in §4.2), where we cast our expressions for J, $\boldsymbol{p}$ and $\boldsymbol{d}$, in terms of Eulerian space–time correlations. So from (2.3), and using (3.5), we have

$$\xi_j[\boldsymbol{x}(t), t]\xi_\ell[\boldsymbol{x}(s), s] = \tfrac{1}{4}\epsilon^2\mathscr{N}_{jl}(y)\mathscr{N}_{\ell r}(y)(G^n_{jkl}(z, t)\exp(\mathrm{i}\varpi^n_{kl}(t)) + \mathrm{c.c.})$$
$$\times(G^m_{\ell qr}(z, s)\exp(\mathrm{i}\varpi^m_{qr}(s)) + \mathrm{c.c.}) \tag{3.8}$$

where $\mathscr{N}_{jl}(y) = \cos(l\beta Y - \delta_{j2}\,\pi/2)$ and, in accord with k, l and n in (3.3b), we set $q, r = 0, \pm 1, \pm 2, \ldots$ and $m = 1, 2, \ldots, \infty$.

The GLM formulation permits any pertinent average, so with no loss of generality we take first a streamwise average over $2\pi/\alpha$. Then, for any k and q, and on setting $s = t + \tau$,

$$\overline{\exp(\mathrm{i}\varpi^n_{kl}(t))\exp(\pm\mathrm{i}\varpi^m_{qr}(s))}^x = \left\{\begin{array}{ll} 1 & \text{for } k = q = 0 \\ \exp(-\mathrm{i}k\alpha U\tau)f(t;\tau) & \text{for } k = \mp q \neq 0 \\ 0 & \text{for } k \neq \mp q \neq 0 \end{array}\right\} \text{for all } m, n, l, r.$$

Further, on allowing for all possible $\omega^0_{kl} = k\alpha c^n_{kl}$ and noting that $c^n_{-kl} = c^n_{kl}$, we see that

$$f(t;\tau) = \exp(-\mathrm{i}k\alpha[(c^n_{kl} - c^m_{\mp kr})t - c^m_{\mp kr}\tau]),$$

so that a subsequent average over time t, with $\tau = 0$, renders $\overline{f(t;\tau)}^t$ zero for all

$k = \mp q \neq 0$ unless $m = n$ and $r = l$. In anticipation of a time average, therefore, we set $m = n$ and $r = l$, although we shall postpone imposing a time average until later. In consequence, and because $G^n_{\ell(-k)(-l)} = G^{*n}_{\ell kl}$, the maximal portion of (3.8) that survives both averages is

$$\overline{\xi_j[x(t),y,z,t]\xi_\ell[x(s),y,z,s]}^x = \tfrac{1}{4}\epsilon^2\mathscr{N}_{jl}(y)\mathscr{N}_{\ell l}(y)\big(G^n_{jkl}(z,t)G^{*n}_{\ell kl}(z,s) \times \exp(-\mathrm{i}k\alpha(U-c^n_{kl})\tau) + \text{c.c.}\big)\ (k,l = 0,\pm1,\pm2,\ldots). \qquad (3.9)$$

Furthermore, the requirement $r = l$ acts to decompose the resulting average into spanwise dependent and spanwise independent parts. This is reflected in the now generalized Jacobian (2.3) which becomes

$$J(y,z,t;\tau) = 1 - \frac{\epsilon^2}{8}\left\{\frac{1}{2}\left[\mathscr{B}_J + \left(\mathscr{B}_J + \frac{2(k\alpha l\beta)^2}{\gamma^2}\mathscr{C}_J\right)\cos 2l\beta Y\right]\exp(-\mathrm{i}\theta^n_{kl}\tau) + \text{c.c.}\right\} \qquad (3.10)$$

where

$$\mathscr{B}_J = k^2\alpha^2(|\phi^n_{kl}|^2 I^n_{kl}(z,t)I^{*n}_{kl}(z,s))'',$$

$$\mathscr{C}_J = \frac{2l^2\beta^2}{\gamma^2}|\phi^{n\,\prime}_{kl} + \mho^n_{kl}|^2 I^n_{kl}(t)I^{*n}_{kl}(s) - ([(|\phi^n_{kl}|^2)' + \phi^n_{kl}\mho^{n\,*}_{kl} + \phi^{n\,*}_{kl}\mho^n_{kl}]I^n_{kl}(z,t)I^{*n}_{kl}(z,s))'$$

and $\theta^n_{kl} = k\alpha(U - c^n_{kl})$.

By entirely similar methods and by making use of the identity $E^n_{\ell kl} = E^{*n}_{\ell(-k)(-l)}$, we use (3.4) and (3.5) to obtain terms of the form

$$\overline{\xi_{j,1}[x(t),y,z,t]\breve{u}_j[x(s),y,z,s]}^x = \tfrac{1}{4}\epsilon^2\mathscr{N}^2_{jl}(y)[\mathrm{i}k\alpha G^n_{jkl}(z,t)E^{*n}_{jkl}(z,s)\exp(-\mathrm{i}\theta^n_{kl}\tau) + \text{c.c.}]$$

and

$$\overline{\xi_{j,3}[x(t),y,z,t]\breve{u}_j[x(s),y,z,s]}^x = \tfrac{1}{4}\epsilon^2\mathscr{N}^2_{jl}(y)[G'^n_{jkl}(z,t)E^{*n}_{jkl}(z,s)\exp(-\mathrm{i}\theta^n_{kl}\tau) + \text{c.c.}]$$

and variants thereof for the pseudomomentum (2.6) and generalized Stokes drift (2.4).

Then on writing

$$p_j = \epsilon^2[P_1, 0, P_3] \quad \text{and} \quad d_j = \epsilon^2[D_1, 0, D_3]$$

and on dropping the subscripts k, l and superscript n to reduce clutter, we obtain generalized expressions for P_j and D_j that are functions of y, z, t and the time separation τ. These take the form:

$$P_j = -\tfrac{1}{4}\left\{\tfrac{1}{2}[\mathscr{B}_j + \mathscr{C}_j + (\mathscr{B}_j - \mathscr{C}_j)\cos 2\beta Y]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\} \quad (j = 1,3), \qquad (3.11a)$$

and

$$P_2 = \tfrac{1}{4}\left\{\tfrac{1}{2}\mathscr{B}_2\beta\sin(2\beta Y)\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\} \qquad (3.11b)$$

where

$$\mathscr{B}_1 = \mathrm{i}\alpha\left[\left(\left|\frac{\alpha^2}{\gamma^2}\phi' - \frac{\beta^2}{\gamma^2}\mho\right|^2 + \alpha^2|\phi|^2\right)I(t)A^*(s) + \mathrm{i}\alpha U'\left(\frac{\alpha^2}{\gamma^2}\phi'\phi^* - \frac{\beta^2}{\gamma^2}\mho\phi^*\right)I(t)I^*(s) + \alpha^2U'^2|\phi|^2K(t)I^*(s) - \mathrm{i}\alpha U'\left(\frac{\alpha^2}{\gamma^2}\phi\phi^{*\prime} - \frac{\beta^2}{\gamma^2}\phi\mho^*\right)K(t)A^*(s)\right],$$

$$\mathscr{B}_2 = \left|\frac{\alpha^2}{\gamma^2}\phi' - \frac{\beta^2}{\gamma^2}\mho\right|^2 I(t)A^*(s)$$
$$-\mathrm{i}\alpha U'\left(\frac{\alpha^2}{\gamma^2}\phi\phi^{*\prime} - \frac{\beta^2}{\gamma^2}\phi\mho^*\right)K(t)A^*(s) + \mathrm{i}\alpha U'\left(\frac{\alpha^2}{\gamma^2}\phi'\phi^* - \frac{\beta^2}{\gamma^2}\mho\phi^*\right)I(t)I^*(s)$$
$$-\frac{\alpha^2\beta^2}{\gamma^4}|\phi' + \mho|^2 I(t)A^*(s) + \alpha^2|\phi|^2(I(t)A^*(s) + U'^2K(t)I^*(s)),$$

$$\mathscr{B}_3 = \left(\frac{\alpha^2}{\gamma^2}\phi' I(z,t) - \frac{\beta^2}{\gamma^2}\mho I(z,t) - \mathrm{i}\alpha U'\phi K(z,t)\right)'\left(\frac{\alpha^2}{\gamma^2}\phi^{*\prime} - \frac{\beta^2}{\gamma^2}\mho^*\right)A^*(s)$$
$$+\alpha^2(\phi I(z,t))'\phi^* A^*(s)$$
$$+\mathrm{i}\alpha U'\left(\frac{\alpha^2}{\gamma^2}\phi' I(z,t) - \frac{\beta^2}{\gamma^2}\mho I(z,t) - \mathrm{i}\alpha U'\phi K(z,t)\right)'\phi^* I^*(z,s),$$

with

$$\mathscr{C}_1 = \frac{\mathrm{i}\alpha^3\beta^2}{\gamma^4}|\phi' + \mho|^2 I(t)A^*(s), \quad \mathscr{C}_3 = \frac{\alpha^2\beta^2}{\gamma^4}((\phi' + \mho)I(t))'(\phi^{*\prime} + \mho^*)A^*(s).$$

While the generalized Stokes drift becomes

$$D_1 = \tfrac{1}{4}\left\{-\tfrac{1}{2}\mathrm{i}\alpha[\mathscr{E}_1 + \mathscr{F}_1 + (\mathscr{F}_1 + \mathscr{F}_2)\cos 2\beta Y]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.12a}$$

$$D_2 = \tfrac{1}{4}\left\{\frac{\alpha^2\beta}{2\gamma^2}\mathscr{E}_2\sin(2\beta Y)\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.12b}$$

$$D_3 = \tfrac{1}{4}\left\{-\tfrac{1}{2}\mathrm{i}\alpha[\mathrm{i}\alpha\phi'\phi^* I(t)A^*(s) + \mathscr{E}_3 + (\mathscr{E}_3 + \mathscr{F}_3)\cos 2\beta Y]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.12c}$$

with

$$\mathscr{E}_1 = \left(\frac{\alpha^2}{\gamma^2}|\phi'|^2 - \frac{\beta^2}{\gamma^2}\phi'\mho^*\right)I(t)A^*(s),$$
$$\mathscr{E}_2 = \left[\frac{\beta^2 - \alpha^2}{\gamma^2}(|\phi'|^2 + \phi'\mho^*) + \frac{2\beta^2}{\gamma^2}(\mho\phi^{*\prime} + |\mho|^2) - \phi\phi^{*\prime\prime} - \phi\mho^{*\prime}\right]I(t)A^*(s)$$
$$+\mathrm{i}\alpha U'(\phi\phi^{*\prime} + \phi\mho^*)K(t)A^*(s),$$
$$\mathscr{E}_3 = \alpha^2 U'|\phi|^2 K(t)A^*(s) + \mathrm{i}\alpha\phi\phi^{*\prime}I(t)A^*(s),$$

while

$$\mathscr{F}_1 = -\mathrm{i}\alpha U'\left(\frac{\alpha^2}{\gamma^2}\phi\phi^{*\prime} - \frac{\beta^2}{\gamma^2}\phi\mho^*\right)K(t)A^*(s)$$
$$+\left(\frac{\alpha^2}{\gamma^2}\phi\phi^{*\prime\prime} - \frac{\beta^2}{\gamma^2}\phi\mho^{*\prime}\right)I(t)A^*(s) + \tfrac{1}{2}\mathrm{i}\alpha U''|\phi|^2 I(t)I^*(s),$$
$$\mathscr{F}_2 = \left(\frac{\alpha^2 - \beta^2}{\gamma^2}\phi' - \frac{2\beta^2}{\gamma^2}\mho\right)\left(\frac{\alpha^2}{\gamma^2}\phi^{*\prime} - \frac{\beta^2}{\gamma^2}\mho^*\right)I(t)A^*(s),$$
$$\mathscr{F}_3 = \mathrm{i}\alpha\left(\frac{\alpha^2 - \beta^2}{\gamma^2}\phi'\phi^* - \frac{2\beta^2}{\gamma^2}\mho\phi^*\right)I(t)A^*(s).$$

Observe that while the streamwise and normal ($j = 1$ and 3) components have the same form as J, i.e. spanwise independent and spanwise dependent parts, the ($j = 2$) components have only the latter and are non-zero only in the presence of oblique modes (i.e. $\beta \neq 0$).

3.2. *Lagrangian integrals*

It remains, of course, to evaluate the integrals (3.6) and (3.7) and the products $I(t)A^*(s)$, $I(t)I^*(s)$, $K(t)A^*(s)$ and $K(t)I^*(s)$ appearing in (3.10) to (3.12). In order to do so it is helpful to first rewrite them in a form that will become ultimately a power series in terms of the small parameter ϵ. We thus assume each $A^n_{kl}(t)$ is continuously differentiable and then integrate by parts, to find

$$I(t) = \exp(-\mathrm{i}\theta t)\left[\left.\frac{\exp(\mathrm{i}\theta\zeta)A}{\mathrm{i}\theta}\right|_{t_0}^{t} - \left.\frac{\exp(\mathrm{i}\theta\zeta)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}A}{\mathrm{d}\zeta}\right|_{t_0}^{t} + \int_{t_0}^{t}\frac{\exp(\mathrm{i}\theta\zeta)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}^2A}{\mathrm{d}\zeta^2}\mathrm{d}\zeta\right], \qquad (3.13)$$

and

$$K(t) = \exp(-\mathrm{i}\theta t)\left[\left.\frac{\exp(\mathrm{i}\theta p)A}{(\mathrm{i}\theta)^2}\right|_{t_0}^{t} - 2\int_{t_0}^{t}\frac{\exp(\mathrm{i}\theta p)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}A}{\mathrm{d}p}\mathrm{d}p + \int_{t_0}^{t}\int_{t_0}^{p}\frac{\exp(\mathrm{i}\theta\zeta)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}^2A}{\mathrm{d}\zeta^2}\mathrm{d}\zeta\,\mathrm{d}p\right]$$
$$- \left.\left(A - \frac{\mathrm{d}A}{\mathrm{d}t}\frac{1}{\mathrm{i}\theta}\right)\right|_{t=t_0}\frac{\exp(\mathrm{i}\theta t_0)}{\mathrm{i}\theta}(t - t_0). \qquad (3.14)$$

Observe that both integrals are oscillatory as $\exp(\mathrm{i}\theta t)$ and that $K(t)$ diverges as $(t - t_0)$ if any A or $\mathrm{d}A/\mathrm{d}t$ at $t = t_0$ is other than zero. Such complications are mathematical artifacts and arise because finite t_0 implies finite wave amplitude, which means that different particles in an averaged ensemble are located on different streamlines. The resolution, as Craik (1982*b*) realized, is to require $t_0 \to -\infty$ so that the integral begins when the waves are infinitesimal. If in the embryonic stages of each wave $\mathrm{Re}\{A\} \propto \exp(\sigma t)$ say, then as $t_0 \to -\infty$ all time derivatives with respect to A, and A itself, are zero. So, on letting $t_0 \to -\infty$ and on writing A in terms of the timescale κ, then what remains of (3.13) and (3.14) is

$$I(t) = \frac{A}{\mathrm{i}\theta} - \frac{\epsilon^\lambda}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}A}{\mathrm{d}\kappa} + \epsilon^{2\lambda}\exp(-\mathrm{i}\theta t)\int_{-\infty}^{t}\frac{\exp(\mathrm{i}\theta\zeta)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}^2A}{\mathrm{d}\kappa^2}\mathrm{d}\zeta, \qquad (3.15)$$

$$K(t) = \frac{A}{(\mathrm{i}\theta)^2} - \frac{2\epsilon^\lambda}{(\mathrm{i}\theta)^3}\frac{\mathrm{d}A}{\mathrm{d}\kappa} + 2\epsilon^{2\lambda}\exp(-\mathrm{i}\theta t)\int_{-\infty}^{t}\frac{\exp(\mathrm{i}\theta p)}{(\mathrm{i}\theta)^3}\frac{\mathrm{d}^2A}{\mathrm{d}\kappa^2}\mathrm{d}p$$
$$+\epsilon^{2\lambda}\exp(-\mathrm{i}\theta t)\int_{-\infty}^{t}\int_{-\infty}^{p}\frac{\exp(\mathrm{i}\theta\zeta)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}^2A}{\mathrm{d}\kappa^2}\mathrm{d}\zeta\,\mathrm{d}p. \qquad (3.16)$$

3.3. *Waves subject to slow modulations in amplitude*

Evaluating the integrals (3.15) and (3.16) is straightforward if $A^n_{kl}(t)$ is known, as would be the case say in a direct simulation of a streamwise periodic wavy shear flow, in which ϕ^n_{kl} and $\mho^n_{kl}$ are used as basis functions. However, before attempting such a calculation it is appropriate to note that wave history is relegated solely to the integral terms and that those terms are $O(\epsilon^{2\lambda})$. Thus, while wave history is crucial to the correct evaluation of the integrals when the waves grow on the same timescale as their period (i.e. $\lambda = 0$), it is far less important for waves growing more slowly ($\lambda \geqslant 1$). Indeed, since a typical product in (3.10)–(3.12) has the form

$$I(t)A^*(s) = \frac{A(t)A^*(s)}{\mathrm{i}\theta} - \epsilon^\lambda\frac{A^*(s)}{(\mathrm{i}\theta)^2}\frac{\mathrm{d}A(t)}{\mathrm{d}\kappa} + O(\epsilon^{2\lambda}) = \frac{A(t)A^*(s)}{\mathrm{i}\vartheta} + O(\epsilon^{2\lambda}) \quad (\lambda \geqslant 1),$$

where $\vartheta_k^n = k\alpha(U - c_k^n - \mathrm{i}c_{\mathrm{i}k}^n) = k\alpha\mathscr{U}_k^n$ with $k\alpha c_i = \epsilon^\lambda A^{-1}\mathrm{d}A/\mathrm{d}\kappa$, that product can be evaluated – provided the waves are growing slowly – solely from the correlation of instantaneous amplitudes at or near the time of interest.

In consequence, we restrict attention to temporal modulations in wave amplitude that are slow compared with the wave period, i.e. $\lambda \geqslant 1$. Then,

$$I(t)A^*(s) \sim \frac{A(t)A^*(s)}{\mathrm{i}\vartheta} + O(\epsilon^{2\lambda}), \quad I(t)I^*(s) \sim \frac{A(t)A^*(s)}{|\vartheta|^2} + O(\epsilon^{2\lambda}), \tag{3.17a}$$

$$K(t)A^*(s) \sim \frac{A(t)A^*(s)}{(\mathrm{i}\vartheta)^2} + O(\epsilon^{2\lambda}), \quad K(t)I^*(s) \sim \frac{A(t)A^*(s)}{\mathrm{i}\vartheta|\vartheta|^2} + O(\epsilon^{2\lambda}), \tag{3.17b}$$

from which we can write (3.10)–(3.12) in terms of the amplitude products.

Then, the Jacobian becomes

$$J = 1 - \frac{\epsilon^2}{8}\left\{\tfrac{1}{2}A(t)A^*(s)\left[\mathscr{G}_J + \left(\mathscr{G}_J + \frac{2\beta^2}{\gamma^2}\mathscr{H}_J\right)\cos 2\beta Y\right]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.18}$$

with

$$\mathscr{G}_J = \left(\frac{|\phi|^2}{|\mathscr{U}|^2}\right)'', \quad \mathscr{H}_J = \frac{2\beta^2}{\gamma^2}\frac{|\phi' + \mho|^2}{|\mathscr{U}|^2} - \left(\frac{\phi\phi^{*\prime} + \phi\mho^*}{|\mathscr{U}|^2}\right)' - \left(\frac{\phi'\phi^* + \phi^*\mho}{|\mathscr{U}|^2}\right)'.$$

Accordingly, from (3.11), the pseudomomentum becomes

$$P_1 = -\frac{1}{4}\left\{\frac{A(t)A^*(s)}{2}\mathscr{U}^*\left[\mathscr{G}_1 + \left(\mathscr{G}_1 - \frac{2\beta^2}{\gamma^2}\mathscr{H}_1\right)\cos 2\beta Y\right]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.19a}$$

$$P_2 = -\frac{1}{4}\left\{\frac{\mathrm{i}\beta\mathscr{U}^*}{2\alpha|\mathscr{U}|^2}A(t)A^*(s)\mathscr{G}_2\sin(2\beta Y)\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.19b}$$

$$P_3 = -\frac{1}{4}\left\{\frac{A(t)A^*(s)}{2\mathrm{i}\alpha}\mathscr{U}^*\left[\mathscr{G}_3 + \frac{\beta^2}{\gamma^2}\mathscr{H}_3 + \left(\mathscr{G}_3 - \frac{\beta^2}{\gamma^2}\mathscr{H}_3\right)\cos 2\beta Y\right]\right.$$
$$\left.\times\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.19c}$$

where

$$\mathscr{G}_1 = \frac{\alpha^2}{\gamma^2}\left|\left(\frac{\phi}{\mathscr{U}}\right)'\right|^2 + \frac{\beta^2}{\gamma^2}\left|\frac{\mho}{\mathscr{U}} + \frac{\mathscr{U}'\phi}{\mathscr{U}^2}\right|^2 + \alpha^2\left|\frac{\phi}{\mathscr{U}}\right|^2, \quad \mathscr{H}_1 = \frac{\alpha^2}{\gamma^2}\left(\left|\frac{\phi'}{\mathscr{U}}\right|^2 + \left|\frac{\mho}{\mathscr{U}}\right|^2\right),$$

$$\mathscr{G}_2 = \left|\frac{\alpha^2}{\gamma^2}\phi' - \frac{\beta^2}{\gamma^2}\mho\right|^2$$
$$-\frac{\mathscr{U}'(\mathscr{U} + \mathscr{U}^*)}{|\mathscr{U}|^2}\left(\frac{\alpha^2}{\gamma^2}\phi\phi^{*\prime} - \frac{\beta^2}{\gamma^2}\phi\mho^*\right) - \frac{\alpha^2\beta^2}{\gamma^4}|\phi' + \mho|^2 + \alpha^2|\phi|^2\left(1 + \frac{\mathscr{U}'^2}{\alpha^2|\mathscr{U}|^2}\right),$$

$$\mathscr{G}_3 = \left(\frac{\alpha^2}{\gamma^2}\frac{\phi'}{\mathscr{U}} - \frac{\beta^2}{\gamma^2}\frac{\mho}{\mathscr{U}} - \frac{\mathscr{U}'}{\mathscr{U}^2}\phi\right)'\left(\frac{\alpha^2}{\gamma^2}\frac{\phi^{*\prime}}{\mathscr{U}^*} - \frac{\beta^2}{\gamma^2}\frac{\mho^*}{\mathscr{U}^*} - \frac{\mathscr{U}'}{\mathscr{U}^{*2}}\phi^*\right) + \alpha^2\left(\frac{\phi}{\mathscr{U}}\right)'\frac{\phi^*}{\mathscr{U}^*},$$

$$\mathscr{H}_3 = \frac{\alpha^2}{\gamma^2}\left(\frac{\phi' + \mho}{\mathscr{U}}\right)'\left(\frac{\phi^* + \mho^*}{\mathscr{U}^*}\right).$$

While from (3.12), the generalized Stokes drift becomes

$$D_1 = \frac{1}{4}\left\{\tfrac{1}{2}A(t)A^*(s)\left[\mathscr{I}_1 + \left(\mathscr{I}_1 + \frac{2\beta^2}{\gamma^2}\mathscr{J}_1\right)\cos 2\beta Y\right]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.20a}$$

$$D_2 = \frac{1}{4}\left\{\frac{\mathrm{i}\alpha\beta}{2\gamma^2}A(t)A^*(s)\mathscr{I}_2\sin(2\beta Y)\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.20b}$$

$$D_3 = \frac{1}{4}\left\{-\tfrac{1}{2}\mathrm{i}\alpha A(t)A^*(s)\left[\mathscr{I}_3 + \left(\mathscr{I}_3 - \frac{2\beta^2}{\gamma^2}\mathscr{J}_3\right)\cos 2\beta Y\right]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}, \tag{3.20c}$$

where

$$\mathscr{I}_1 = -\frac{\alpha^2}{\gamma^2}\left(\frac{\phi\phi^{*\prime}}{\mathscr{U}}\right)' + \frac{\beta^2}{\gamma^2}\left(\frac{\phi\mho^*}{\mathscr{U}}\right)' + \frac{\mathscr{U}''|\phi|^2}{2|\mathscr{U}|^2},$$

$$\mathscr{J}_1 = \frac{\alpha^2}{\gamma^2}\frac{|\phi'|^2}{\mathscr{U}} - \frac{\beta^2}{\gamma^2}\frac{\phi'\mho^*}{\mathscr{U}} + \frac{\alpha^2}{\gamma^2}\frac{\mho\phi^{*\prime}}{\mathscr{U}} - \frac{\beta^2}{\gamma^2}\frac{|\mho|^2}{\mathscr{U}},$$

$$\mathscr{I}_2 = \left(\frac{\phi\phi^{*\prime}}{\mathscr{U}}\right)' - \left(\frac{\phi\mho^*}{\mathscr{U}}\right)' - \frac{2\beta^2}{\gamma^2\mathscr{U}}|\phi' + \mho|^2,$$

$$\mathscr{I}_3 = \left(\frac{\mathscr{U}^*|\phi|^2}{|\mathscr{U}|^2}\right)', \quad \mathscr{J}_3 = \frac{\mathscr{U}^*}{|\mathscr{U}|^2}(\phi'\phi^* + \mho\phi^*).$$

Remember that (3.18)–(3.20) are generalizations of J etc. (which we require in § 4.2), and that if we wish to use this form directly, we must set τ to zero; each measure is then a function solely of y, z and t. Furthermore, remember that the amplitudes and eigenfunctions are summed as $A^n_{kl}(t)A^{*n}_{kl}(s)$ and $\phi^n_{kl}(z)\phi^{*n}_{kl}$ over the relevant range of n, k and l. As mentioned above, this is straightforward when $A^n_{kl}(t)$ are known from a direct simulation, but of particular interest is to determine $\boldsymbol{p}$ and $\boldsymbol{d}$ from correlations that are measurable physically, and we shall discuss doing so in § 4.

Before doing so, however, we note that with $k = n = 1$, $\beta = \tau = 0$ and t sufficiently small that $\alpha c_i = \mathrm{Im}\{\omega^1_{10}\}$, then $\mathrm{Re}\{A\} = \exp(\alpha c_i t)$ and equations (3.19)–(3.20) reduce to their two-dimensional counterparts given by Craik (1982*b*; his (3.3) and (3.4)) and (3.18) by Phillips (1998*a*; his (5.15)). Accordingly, with the further restrictions that $U = \mho = 0$ and $\phi = \mathrm{e}^{\varrho z}$ but $\beta \neq 0$, we recover Craik & Leibovich's (1976) D_1 result (their (59)) for a discrete spectrum of irrotational oblique wave pairs of equal amplitude.

3.4. *Phase mixing*

As noted above (see (3.10)–(3.12)), J and the streamwise and vertical components of P_j and D_j are each composed of spanwise independent and spanwise dependent parts, while P_2 and D_2 are strictly spanwise dependent. However, although spanwise dependence at each $l\beta$ is reflected in terms of the form $l^2\beta^2\cos[2l\beta(y+y_l)-\delta_{j2}\,\pi/2]$, the overall dependence is subject to phase mixing between various Fourier components. Indeed, if a fixed amount of wave energy is distributed between M discrete oblique wave pairs of random phase, the nonlinear measures become more nearly uniform spanwise as M is increased and depict no spanwise dependence in the limit $M \to \infty$ (Craik & Leibovich 1976). In this same limit $P_2, D_2 \to 0$; but this result does not mean the surviving portions of P_j and D_j are due exclusively to two-dimensional waves within the spectrum, rather oblique components also contribute to the spanwise independent part, as we see (for example) in $\mathscr{G}_1$.

4. Measurable quantities

4.1. *Interfacial*

Two different Eulerian correlations are relevant to the evaluation of (3.18)–(3.20). The first pertains to surface waves where it is possible to measure the directional instantaneous wave slope

$$\boldsymbol{\varepsilon}(t) = \tfrac{1}{2}\{\mathscr{A}_{kl}(t)\exp(-\mathrm{i}\omega_{kl}t)[k\alpha\exp(\mathrm{i}k\alpha x), l\beta\exp(\mathrm{i}l\beta y)] + \text{c.c.}\},$$

at some point (x, y) say, where $\mathscr{A}_{kl}(t)$ is the amplitude of waves with wavenumbers $k\alpha$, $l\beta$ at time t, and ω_{kl} is real. Knowledge of $\boldsymbol{\varepsilon}(t)$ then leads to the autocorrelation

$$\overline{\varepsilon_i(x, y, t)\varepsilon_i(x, y, t)} = \tfrac{1}{4}\{(\delta_{i1}k^2\alpha^2 + \delta_{2i}l^2\beta^2)\mathscr{A}_{kl}(t)\mathscr{A}^*{}_{kl}(t) + \text{c.c}\} \quad (i = 1, 2), \tag{4.1}$$

either through a time (Bock & Hara 1995; Hara, Bock & Donelan 1997) or ensemble average (Melville, Shear & Veron 1998) and ultimately the frequency–wavenumber slope spectrum and thus $\mathscr{A}_{kl}$, from which we can extract A^n_{kl} as

$$\mathscr{A}_{kl}(t)\exp(-\mathrm{i}\omega_{kl}t) + \text{c.c.} = \epsilon A^n_{kl}(t)\phi^n_{kl}(0)\exp(-\mathrm{i}\omega^0_{kl}t) + \text{c.c.} \tag{4.2}$$

But to proceed we need $\phi^n_{kl}(z)$ and that must be found by solving the relevant linear eigenvalue problem given $U(z)$ and the boundary conditions for the case at hand; see §7. Of course, if the waves are irrotational, the process is somewhat simpler and such an example is given by Phillips (2000*b*) using the continuous slope spectrum Smith (1992) measured in the Pacific ocean.

4.2. *Interior*

The second measurable correlation pertains to the interior of the flow and follows from the fluctuating velocity field in the form of space–time correlations Q_{ij} and $\mathscr{Q}_{jlk}$, defined by

$$\begin{aligned}\epsilon^2 Q_{jl}(y, z, t; U\tau, 0, 0, \tau) &= \overline{\breve{u}_j(x(t), y, z, t)\breve{u}_l(x(s), y, z, s)}^x \\ &= \tfrac{1}{4}\epsilon^2\{E^n_{jk}E^{*n}_{lk}\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\}\end{aligned} \tag{4.3}$$

and

$$\epsilon^2\mathscr{Q}_{jlk}(y, z, t; U\tau, 0, 0, \tau) = \overline{\breve{u}_{j,k}(x(t), y, z, t)\breve{u}_l(x(s), y, z, s)}^x.$$

Here, the time separation is τ and the spatial separation is $r = U\tau$. Observe that (4.3) yields products as $A(t)A^*(s)$, so that by suitable manipulation, namely, differentiation with respect to x (prior to averaging) or integration with respect to τ (after averaging), we can reproduce each of the components in (3.18)–(3.20) in terms of Q_{ij}. Furthermore, because (3.18)–(3.20) assume slowly growing waves, it is consistent (to leading order) to ignore any variation in A due to t when integrating with respect to τ; thus, $A(t)A^*(s) \equiv A^*(t)A(s) \equiv A^*(t)A(t)$ and $\theta \equiv \alpha\mathscr{U}$. So, for example,

$$\int_{\zeta_0}^{\zeta}\overline{\breve{u}_{1,1}(t)\breve{u}_1(s)}^x\mathrm{d}\tau = -\frac{\epsilon^2}{4}\left\{A(t)A^*(t)\frac{1}{2\mathscr{U}}\left|\frac{\alpha^2}{\gamma^2}\phi' - \frac{\beta^2}{\gamma^2}\mho\right|^2(1 + \cos 2\beta Y) \times\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\right\}\Bigg|_{\zeta_0}^{\zeta},$$

which recovers the first term in P_1 (see (3.11*a*) and $\mathscr{B}_1$). Note, too, that the space–time correlation includes not only spanwise variations for all $l\beta$, but contributions those same $l\beta$ make to the spanwise independent part.

It is now evident why we sought the generalized form introduced in §3.1; but this process also introduces double and triple integrals which, as we saw in § 3.2, can introduce spurious divergent terms. To exclude such terms we must carefully determine the limits of integration and this is best done by comparing the integral form with a known solution to its counterpart in § 3.3.

Consider then the Jacobian (3.18), whose term $|\phi|^2/|\mathscr{U}|^2$ is approximated by the double integral

$$\int_{\kappa^\star}^{\kappa_0}\int_{\zeta_0}^{\zeta} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta = \frac{\epsilon^2}{4}\left\{\tfrac{1}{2}\alpha^2 A(t)A^*(t)|\phi|^2(1+\cos 2\beta Y)\int_{\kappa^\star}^{\kappa_0}\int_{\zeta_0}^{\zeta}\exp(-\mathrm{i}\theta\tau)\mathrm{d}\tau\mathrm{d}\zeta + \mathrm{c.c.}\right\}, \tag{4.4}$$

and restrict attention to monochromatic plane waves, so $k = n = 1$ and $l = 0$. On integrating the right-hand side of (4.4) we see (i), that divergent terms are excluded provided $\theta\zeta_0 = \pm N\pi$ ($N = 0, 1, 2, \ldots$) and (ii), that the first term in (3.18) (with $\tau = 0$) is recovered provided the limit $\kappa_0 = \zeta_0$ and the limit $\theta\kappa^\star = \pm(N+1)\pi/2$. Of course, to expedite the calculation, it is prudent to confine attention to $\theta\kappa^\star > 0$ and restrict N to $N = 0$. Then the inner integral is evaluated from $\zeta_0 = 0$ to ζ and the outer integral from $\kappa^\star$ to 0. In short, $\kappa^\star$ is chosen to ensure the value of the double integral is $-\theta^{-2}$. Indeed, in general, we require all integrals to have the value $\operatorname{sgn}\theta^{-m}$, where m is the degree of integration.

Of course there must also be a $\theta_k^n\kappa^{\star n}_{\ k} = \frac{1}{2}\pi$ synonymous with the first zero of the double integral for each k, l and n in a spectrum of waves; but this is not immediately helpful given Q_{ij} for the spectrum with the intent to proceed numerically. In this instance, we determine $\kappa^\star$ as follows.

We first non-dimensionalize Q_{ij} and τ as $R_{ij} = Q_{ij}/\overline{\breve{u}_i\breve{u}_j}$ and $\eta = \tau\theta$. Then, with no loss of generality and in accord with our findings above, define θ by the requirement

$$R_{33}|_{\eta=1} = \tfrac{1}{2}. \tag{4.5}$$

Next, we note that

$$\int_0^{\kappa^\star} Q_{33}\mathrm{d}\tau = \frac{\overline{\breve{u}_3\breve{u}_3}}{\theta}\int_0^{\eta^\star} R_{33}\mathrm{d}\eta$$

and that to concur with our findings above we must define $\eta^\star = \kappa^\star\theta$ by the constraint

$$\int_0^{\eta^\star} R_{33}\mathrm{d}\eta = 1. \tag{4.6}$$

Lastly, two further conditions are necessary to proceed: (i) that the double integral have its first zero at $\eta = \eta^\star$, and (ii) that the triple integral have its first zero at $\eta = 0$. These requirements are satisfied by noting the class of kernel, i.e. even or odd, and by appropriately ordering the integration. Examples using this procedure are given in § 6. Finally, since $\mathrm{Re}\{\mathrm{i}\alpha\mathscr{U}^*\} = -A^{-1}\mathrm{d}A/\mathrm{d}t$, we rewrite those components premultiplied by i, e.g. P_3 and D_3, in terms of $A^*\mathrm{d}A/\mathrm{d}t$ and recover them by taking the derivative of Q_{ij} with respect to time (see (4.10) and (4.13)).

Having learned how to determine $\kappa^\star$ and evaluate our multiple integrals, we now return to our nonlinear measures in integral form. Here to $O(\epsilon^2)$, the Jacobian is

$$J(y,z,t) = 1 + \frac{1}{2}\left[\frac{\partial^2}{\partial y^2}\int_{\kappa^\star}^{0}\int_{0}^{\zeta} Q_{22}\mathrm{d}\tau\mathrm{d}\zeta + \frac{\partial^2}{\partial y\partial z}\int_{\kappa^\star}^{0}\int_{0}^{\zeta}(Q_{32}+Q_{23})\mathrm{d}\tau\mathrm{d}\zeta + \frac{\partial^2}{\partial z^2}\int_{\kappa^\star}^{0}\int_{0}^{\zeta} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta\right], \tag{4.7}$$

while, on noting

$$\int_{\zeta_0}^{\zeta} \overline{\breve{u}_{i,1}(t)\breve{u}_j(s)}^x \mathrm{d}\tau = -\int_{\zeta_0}^{\zeta} \frac{\partial}{\partial r}\overline{\breve{u}_i(t)\breve{u}_j(s)}^x \mathrm{d}\tau = -\epsilon^2 \frac{Q_{ij}}{U}\bigg|_{\zeta_0}^{\zeta},$$

the $O(\epsilon^2)$ x-, y- and z-components of the pseudomomentum are

$$P_1(y,z,t) = \int_0^{\kappa^\star} \frac{\partial}{\partial r} Q_{jj}\mathrm{d}\tau + U' \int_{\kappa^\star}^0 \int_0^{\zeta} \frac{\partial}{\partial r}(Q_{31} - Q_{13})\mathrm{d}\tau\mathrm{d}\zeta + U'^2 \int_{\kappa^\star}^0 \int_0^{\chi} \int_{\kappa^\star}^{\zeta} \frac{\partial}{\partial r} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi, \quad (4.8)$$

$$P_2(y,z,t) = \frac{1}{2}\frac{\partial}{\partial t}\left\{\int_{\kappa^\star}^0 \int_0^{\zeta} \mathcal{Q}_{jj2}\mathrm{d}\tau\mathrm{d}\zeta + U' \int_{\kappa^\star}^0 \int_0^{\chi} \int_{\kappa^\star}^{\zeta} (\mathcal{Q}_{312} - \mathcal{Q}_{132})\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi - U'^2 \int_{\kappa^\star}^0 \int_0^{\gamma} \int_{\kappa^\star}^{\chi} \int_0^{\zeta} \mathcal{Q}_{332}\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi\mathrm{d}\gamma\right\} \quad (4.9)$$

and

$$P_3(y,z,t) = \frac{1}{2}\frac{\partial}{\partial t}\left\{\int_{\kappa^\star}^0 \int_0^{\zeta} \mathcal{Q}_{jj3}\mathrm{d}\tau\mathrm{d}\zeta + U' \int_{\kappa^\star}^0 \int_0^{\chi} \int_{\kappa^\star}^{\zeta} (\mathcal{Q}_{313} - \mathcal{Q}_{133})\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi + U'' \int_{\kappa^\star}^0 \int_0^{\chi} \int_{\kappa^\star}^{\zeta} Q_{31}\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi - U'^2 \int_{\kappa^\star}^0 \int_0^{\gamma} \int_{\kappa^\star}^{\chi} \int_0^{\zeta} \mathcal{Q}_{333}\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi\mathrm{d}\gamma - U'U'' \int_{\kappa^\star}^0 \int_0^{\gamma} \int_{\kappa^\star}^{\chi} \int_0^{\zeta} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi\mathrm{d}\gamma\right\}. \quad (4.10)$$

Lastly, the $O(\epsilon^2)$ x-, y- and z-components of the generalized Stokes drift are

$$D_1(y,z,t) = \frac{\partial}{\partial z}\int_0^{\kappa^\star} Q_{31}\mathrm{d}\tau - \tfrac{1}{2}U'' \int_{\kappa^\star}^0 \int_0^{\zeta} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta, \quad (4.11)$$

$$D_2(y,z,t) = -\frac{1}{2}\frac{\partial}{\partial t}\left\{\int_{\kappa^\star}^0 \int_0^{\zeta} \mathcal{Q}_{j2j}\mathrm{d}\tau\mathrm{d}\zeta + U' \int_{\kappa^\star}^0 \int_0^{\chi} \int_{\kappa^\star}^{\zeta} \mathcal{Q}_{321}\mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}\chi\right\} \quad (4.12)$$

and

$$D_3(y,z,t) = \frac{\kappa^\star}{2}\frac{\partial \mathscr{D}_3}{\partial t} \quad \text{where} \quad \kappa^\star \mathscr{D}_3 = -\frac{\partial}{\partial z}\int_{\kappa^\star}^0 \int_0^{\zeta} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta. \quad (4.13)$$

As a check we return to our example with $k = n = 1$; then from (4.3) with $\beta = 0$ we have

$$Q_{11}, Q_{13}, Q_{33} = \tfrac{1}{4}\epsilon^2\{A(t)A^*(t)\left[|\phi'|^2, \mathrm{i}k\alpha\phi'\phi^*, k^2\alpha^2|\phi|^2\right]\exp(-\mathrm{i}\theta\tau) + \text{c.c.}\},$$

which, when substituted into our integral equations (4.7) to (4.13), recover (3.18)–(3.20) with $\tau = 0$. Then, on setting $A(t) = \mathrm{Re}\{\exp(\alpha c_i t)\}$, we recover Craik's (1982*b*) expressions for D_j and P_j ($j = 1, 3$) as before.

5. Wave fields with a continuous spectrum of wavenumbers

Consider now a three-dimensional wave field composed of symmetric wave pairs with amplitudes that may vary from pair to pair, subject to the boundary condition

$|H| \to \infty$ (see § 3). Then, the waves comprise a continuous spectrum which, we assume, has finite total energy and Fourier components with random phases. Our intent, as above, is to express p_j and d_j in terms of space–time correlations.

The ensuing analysis will, of course, mimic our work above, but with summations replaced by integrals over wave space. However, since (4.7)–(4.13) are devoid of explicit summation over wave space and streamwise periodicity, they alone must carry over to the case of a continuous non-periodic spectrum. Furthermore, phase mixing (see § 3.4) necessitates that spanwise variations in P_j and D_j have statistically zero variance and thus no structure, leaving, in essence, a two-dimensional rectified second-order field. Thus, J simplifies to

$$J(z,t) = 1 + \frac{1}{2}\frac{\partial^2}{\partial z^2}\int_{\kappa^\star}^{0}\int_{0}^{\zeta} Q_{33}\mathrm{d}\tau\mathrm{d}\zeta,$$

and $P_2, D_2 \to 0$. The remaining expressions (4.8)–(4.13) are essentially unchanged except that they are now solely functions of z and t.

Hence, our object would appear complete in that we have obtained expressions for J, P_j and D_j that are consistent for discrete and continuous spectrums of rotational waves and which are expressed in terms of measurable quantities. However, previous attempts to express D_i and P_i in terms of space–time correlations are at variance with (4.7) through (4.13).

5.1. *Previous attempts*

Attempts to express D_j and P_j in terms of space–time correlations date from Lumley (1986), Phillips (1988) and Leibovich (1992). Lumley took an Eulerian approach and confined attention to D_j, while Phillips and Leibovich dealt with GLM to derive P_j; Leibovich also gives a cursory overview of the derivation whereas Phillips (1988, 1991) simply states the results. Although their limits of integration are undefined, their lead terms for P_1 and D_1 concur with (4.8) and (4.11) but their additive terms, i.e. those multiplied by U' and U'', do not. To conclude the present work, therefore, we must resolve the variance.

We find that the variance is, in essence, due to the choice of t_0. Recall that values other than the limit $t_0 \to -\infty$ ensure divergent behaviour (see § 3.2) and that while we use the appropriate limit, previous authors set t_0 to zero. Furthermore, consistent with finite t_0, they are able to cast (2.8) into the form given by Leibovich (1992; his (30)), which leads to very different results for the multiple integrals and thus the variance.

6. An example

As an example, we consider constant-mass-flux plane channel-flow subject to a discrete spectrum of two- and three-dimensional progressive waves; and, in particular, the flow used by Kim, Moin & Moser (1987) to simulate low-Reynolds-number turbulent channel flow. Here, discrete spectral techniques were used to approximate the Navier–Stokes equation under the assumption that the flow is streamwise and spanwise periodic. Furthermore, the chosen wave spectrum was sufficiently large (192-streamwise × 129-spanwise modes) for spanwise variations to phase mix to almost zero (see § 3.4). Reynolds numbers, based on channel half width and centreline (friction) velocity were 3260 (180). Finally, the calculation was continued over sufficient time for credible statistics to be obtained and this enabled Kim & Hussain (1993) to calculate space–time correlations, which Phillips (2000*a*) later modelled.

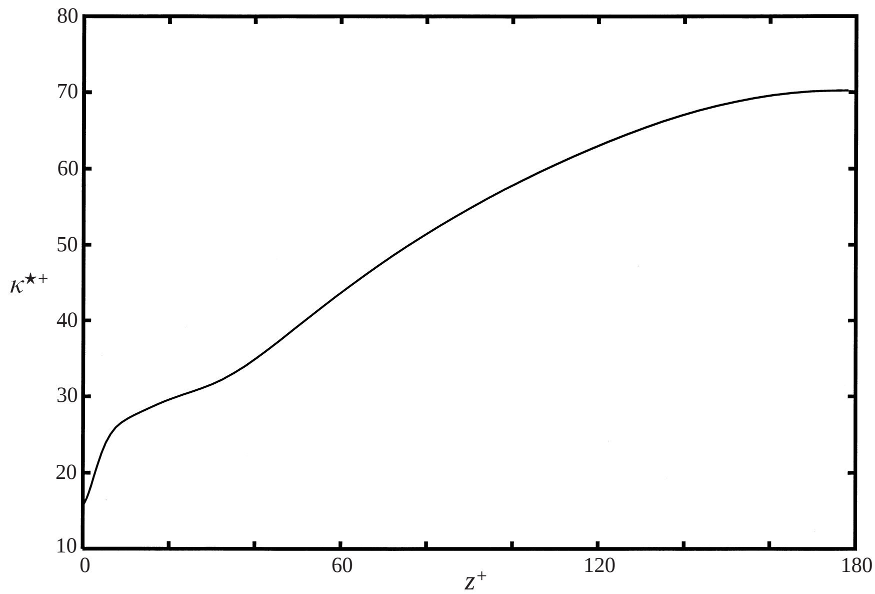

FIGURE 1. The time $\kappa^\star$ with distance from the wall. Both variables are in wall units: $\kappa^{\star +} = \kappa^\star U_\tau^2/\nu$ and $z^+ = zU_\tau/\nu$. The channel centreline is at $z^+ = 180$.

Phillips' model is based upon the Kovasznay–Corrsin conjecture modified for shear flows, and reduces, for the correlations defined by (4.3), to $Q_{ij} = \overline{\breve{u}_i\breve{u}_j}\ \mathscr{R}(\eta)$, where

$$\mathscr{R}(\eta) = (1 + \mathscr{F}(\eta))^{-3/2}, \quad \mathscr{F}(\eta) = \eta^2(1 + B\eta^2)^{-1/2}, \tag{6.1}$$

with B a constant defined by the requirement $\mathscr{R}(1) = \frac{1}{2}$. Since the model is continuous in space and time, it is ideal for our purposes and we employ it to calculate measures mandatory to the GLM description of plane wavy shear flows, namely, J, P_1, D_1 and $\mathscr{D}_3$. The same measures are also crucial to related studies of the instability of the flow to streamwise vortices and the subsequent dynamical behaviour of the said vortices (Phillips 1998*a*, *b*).

6.1. *Non-periodic space–time correlations*

The class of space–time correlations defined by (6.1) is noticeably different from what we studied in §4.2, and before proceeding it is appropriate to discuss this class, beginning with the simpler form

$$\mathscr{R}(\eta) = (1 + C\eta)^{-3/2} \quad (\eta \geqslant 0), \tag{6.2}$$

which is readily integrable. Here, the ensuing integrals are not oscillatory: rather the first is bounded while the double and triple integrals diverge. Of interest, however, is whether (6.2) is an admissible form *vis à vis* the constraints (4.5) and (4.6) and the conditions stated in §4.2. In fact it is; from (4.5) we find that $C \approx 0.5874$ and from (4.6) that $\eta^\star \approx 1.7102$, at which point ($\eta = \eta^\star$) the double integral is necessarily zero (first condition). Furthermore, the triple integral is zero at $\eta = 0$ (second condition). In short, generic variants of (6.1) may be used to evaluate (4.7)–(4.13).

Returning now to (6.1), we find $B \approx 1.8982$ and that $\eta^\star = \kappa^\star\theta \approx 1.5818$. Of course, knowledge of $\overline{\breve{u}_i\breve{u}_j}$ is necessary to deduce θ and $\kappa^\star$ which both vary with z, and that variation ($\kappa^\star$ in wall units) is sketched in figure 1. The Jacobian J is plotted in figure 2. Note that J remains non-zero as it must for the mapping from the true Lagrangian

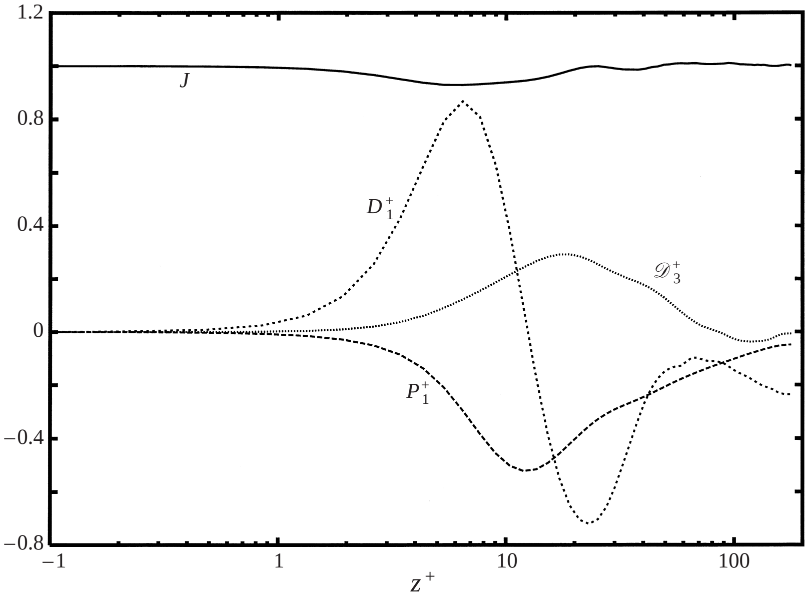

FIGURE 2. The Jacobian and components of the generalized Stokes drift and the pseudomomentum (relative to U_τ) with distance from the wall $z^+ = zU_\tau/\nu$ in a discrete spectrum of progressive waves in plane channel flow. The flow is that of Kim *et al.* (1987) which models low-Reynolds-number turbulent channel flow. The channel centreline is at $z^+ = 180$ and the mean velocity there is $U/U_\tau = 18.3$.

to the generalized Lagrangian mean to remain invertible. The components P_1, D_1 and $\mathscr{D}_3$ relative to the friction velocity $U_\tau = (\nu U'|_{z=0})^{1/2}$ are also plotted in figure 2.

6.2. *Results*

Observe that P_1, which is negative throughout the domain, mimics its counterparts in generic studies of inviscid wavy shear flows (Craik 1982*c*; Phillips & Wu 1994; Phillips & Shen 1996). However, while the inviscid case depicts a singularity at the boundary ($z = 0$), viscosity here enters to bring P_1, D_1 and $\mathscr{D}_3$ to zero. Moreover, although $P_1 \neq D_1$ (because the wave field is rotational), the two components share the same sign over much of the layer. Indeed, the Lagrangian-mean or net mass transport velocity exceeds $U(z)$ only in the wall region, in accord with the convection velocity deduced from space–time correlations both experimentally (Kreplin & Eckelmann 1979) and numerically (Kim & Hussain 1993). On the other hand, $\mathscr{D}_3$ is positive (negative) in the inner region if the wave field if growing (decaying) and changes sign in the logarithmic (i.e. overlap) region. In a scenario where the mean wave field cyclically grows and decays, therefore, we should expect a cyclic mass transfer from, and then towards, the wall.

As for magnitudes, we see that P_1 has a peak value of about $\frac{1}{2}U_\tau$, while D_1 varies by almost $2U_\tau$. Interestingly D_1 changes sign about 12.5 viscous units from the wall, a point coincident with the minimum in P_1 and the point at which turbulence production is a maximum. It is also the point at which D_1' is a minimum and occurs near the centre of a thin (15 viscous units) layer in which $D_1' < 0$. This layer of highly sheared Stokes drift is at the heart of the wall region, long known for its streaks and streamwise vortices, which are thought to play a crucial role in the generation and regeneration of turbulence. Such structures are, of course, secondary to the primary (spanwise independent) Lagrangian mean flow exposed in this example, but it does not seem unreasonable to speculate that the narrow layer of $D_1' < 0$ and its coincidence with the minimum in P_1, are together pivotal to the (currently unknown) nonlinear instability mechanism which spawns wall-layer structures.

7. Discussion

Although we have expressed J, P_j and D_j in terms of measurable Eulerian quantities, the path to their evaluation is not always straightforward. For example, suppose we require P_j and D_j in the water beneath wind-driven surface waves. Here (4.1) is measurable, but in order to evaluate (4.2) we require ϕ_{kl}^n (and possibly $\mho_{kl}^n$) which must be found by solving the linear eigenvalue problem defined by the coupled air–water problem. This was done by Morland & Saffman (1993), but there is a problem. As is the case in such stability problems, the phase velocities of growing waves are subject to a circle theorem, which, in this instance, decrees that it (the phase velocity) should fall between the maximum air- and minimum water-velocity. This necessitates critical layers, and the mapping upon which GLM is based breaks down in the vicinity of critical layers (at least in a discrete spectrum of waves; see §2.2 and Phillips 1998). Laboratory data (Melville *et al.* 1998) of growing wind-driven surface waves indicate that critical layer(s) occur only in the air, so that GLM can be meaningfully applied to events in the water. Of course, whether this is a general result for wind-driven waves is unclear; rather the point to note is that each calculation must be taken on a case by case basis.

Alternatively, we can evaluate J, P_j and D_j in the interior from knowledge of Eulerian space–time correlations, as was done in our examples in §6. This approach can be applied to both discrete and continuous spectrums of waves provided J is non-zero, as discussed in §2. However, of particular interest is whether (4.7)–(4.13) carry over to flows which violate, at some wavenumbers, the assumptions we have invoked.

In deriving (3.18)–(3.20), we assumed waves with amplitudes that grow on a timescale significantly greater than the wave period ($\lambda \geqslant 1$). Equations (3.18)–(3.20) were then expressed (as (4.7)–(4.13)) in terms of velocity correlations, which are, in essence, measures of the kinetic energy of velocity fluctuations. Two features of these correlations are of interest: the first is that they are dominated by that portion of the frequency–wavenumber spectrum which is most energetic; the second is that the rectification process inherent in realizing the correlations acts to suppress less energetic high-frequency high-wavenumber components of the spectrum. Thus, provided that the most energetic fluctuating components of the flow satisfy our assumptions, at least on average, there would seem to be a reasonable case to employ (4.7)–(4.13), even though other portions of the frequency–wavenumber spectrum violate the assumptions.

Prima facie members of this class of flows are those subjected to wave forcing at wavenumbers noticeably smaller than those dominant in the unforced flow. For example, turbulent boundary-layer flow over rigid wavy walls of small amplitude (Phillips *et al.* 1996) and the turbulent shear flow (in both the water and the air) associated with wind-driven surface waves (Phillips, Wu & Jahnke 1999).

However, from a strictly *de rigueur* viewpoint the present analysis is concerned with small-amplitude waves. Of course, a strength of GLM is that theories and flow equations can be derived in its setting which are exact for finite-amplitude waves, so it is pertinent to ask to what point our analysis is valid for finite-amplitude waves, or, more precisely, waves with $O(1)$ slope. The answer is (2.2) because (2.4), (2.6) and (2.8) for p_l, d_l and ξ_l each assume convergent expansions in terms of wave slope. However, the techniques employed in § 3, at least until § 3.3, would carry over to the larger-amplitude case, provided we could credibly evaluate (2.7) for the displacement field, and thence deduce the Lagrangian velocity perturbation, which would together yield the pseudomomentum (2.2).

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